The required number of wide bore cavities for PIPII

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Abstract

We want to calculate the required number of wide bore cavities for operating Booster in the PIPII era. In this era, the beam intensity will be 50% higher than in PIP. We will show that the higher beam current will cause Robinson instability unless beam loading compensation is installed. We will show with ESME simulations that when we have perfect beam loading compensation and with space charge turned on, the required cavity voltage to accelerate the beam from injection to extraction without loss is 1.2 MV. We have also used a simple analytic model to calculate the RF voltage curve and found the same required voltage of 1.2 MV as well. At this voltage, we will show that we will require at least 16 wide bore cavities to be able to operate in the scenario where there is one failed wide bore cavity. In this scenario, we will need to assume that the legacy cavities can operate at 50 kV and the wide bore cavities at 60 kV. IMPORTANT! The 1.2 MV, and 16 cavities quoted here are with caveats that will be discussed in the Introduction.

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I. INTRODUCTION

The goal of the paper is to pin down the following for running Booster in the PIPII era:

1. The minimum required number of wide bore cavities to achieve the necessary accelerating voltage. See section I A to see what we have not covered.

2. The required peak voltage for the legacy cavities.

3. The necessity for beam loading compensation.

We will divide the paper into two parts: the first part will be analytic solutions for the zero beam current scenario that will give us the ballpark estimates to the above requirements. In the second part, we will use ESME to model the Booster beam as a space charge dominated system. Perfect beam loading compensation is simulated by zeroing out the cavity impedances. Our ESME results will be used to estimate the above requirements as well.

The Booster parameters used in this paper are in Appendix C.

In Appendix D, we will show that the theoretical voltage ramp where the bucket contains all the beam for the ramp has the same maximum required voltage found by ESME.

A. Not covered

We did not cover the following:

1. Any phase errors between the cavities that will require more volts to achieve the required accelerating voltage. From our experience in phasing the cavities with beam we have found that we can phase the cavities only to the $\pm 5°$ level. Assuming that this is the sigma value, i.e. $\sigma = 5\pi/180$, then we can apply the formula given in Eq. B7 for 22 RF cavities to get an "effective" reduction in the number of cavities because of this error: $\sqrt{|V_c|^2}/V_p = 21.9$. Thus, the phase error makes it look like we have reduced the number of cavities from 22 to 21.9 or lowered by 0.4% the accelerating voltage. Therefore, small phase errors between cavities should not have a significant impact on the total cavity voltage.

2. Quadrupole dampers traditionally uses amplitude modulation to damp quadrupole oscillations after transition. In PIP, about 8% over the accelerating voltage at transition is necessary for this purpose. The additional volts used for modulation is not covered in this paper.

3. In our ESME simulations, we have assumed PIPII beam parameters. However, if in a working machine, the longitudinal emittance is larger than anticipated, we may require > 1.2 MV at the peak of the RF voltage ramp shown in Fig. 4.

II. REQUIRED ACCELERATING VOLTAGE FOR ZERO CURRENT BEAM

The required accelerating voltage is found using Eq. A9. However, in order to use this formula, we need to know the revolution frequency of the δ-function bunch when the rate of change of momentum is at its maximum, i.e. at $(dp/dt)_{\text{max}}$. From Eq. A8, the momentum at this point is

$$p_* = \frac{p_f + p_i}{2} = \frac{1.463 + 8.889}{2} = 5.18 \text{ GeV/c} \tag{1}$$

This gives a relativistic $\beta = 0.984$ and $f_{\text{rev}}(p_*) = 622.1$ kHz.

When we apply the above values and the synchronous phase $\phi_s = (180 - 50)^\circ = 130^\circ$ (because we are above transition at p_*. See section III), we find that the required accelerating voltage is

$$\begin{aligned} V_{\text{req}} &= \frac{f_{\text{ramp}}(p_f^2 - p_i^2)\pi}{f_{\text{rev}}(p_*)\left[4m^2 + (p_f + p_i)^2\right]^{1/2} \sin\phi} \\ &= \frac{20[\text{Hz}](8.889^2 - 1.463^2)[(\text{GeV/c})^2]\pi}{622.1[\text{kHz}](4 \times (0.9383[\text{GeV/c}^2])^2 + (8.889 + 1.463)^2[(\text{GeV/c})^2])^{1/2} \sin(130\pi/180)} \\ &= 963.4 \text{ kV} \end{aligned} \tag{2}$$

If we assume that V_{req} is equally divided among $N_c = 22$ cavities, then the voltage on each cavity is $V_c = V_{\text{req}}/N_c = 43.8$ kV per cavity.

1. The above result assumes that there is no beam loading, i.e. the required voltage is calculated for a zero current beam.

2. During present PIP operations, each cavity operates at $V_c' = 45$ kV. Likewise, we will suppose that during normal PIPII operations, all the cavities will also operate at

$V_c'(> V_c)$. Therefore, during PIPII, the total operational voltage, $V_{\rm op} = V_c' \times N_c = 45\ [\rm kV] \times 22 = 990$ kV.

A. Required number of cavities for zero beam loading

At present (PIP), the legacy cavities routinely operate at 45 kV. In PIPII, it is assumed that the legacy cavities can operate at 50 kV so that there is some overhead (see sections II A 1 and II A 2 below). However, as of the end of 2019, we have yet to demonstrate that the legacy cavities can operate at 50 kV. Furthermore, in the present PIPII proposal (Dec 2019), there will be $N_{wc} = 6$[1] new wide bore cavities that are assumed to be able to run at 60 kV. Note: as of 2019, the wide bore test cavity has not been demonstrated to work at 60 kV. It has not been installed in Booster because its tuners have to be modified.

The goal is to keep the operational voltage at $V_{\rm op} = 990$ kV even when we lose cavities. The following sections will tell us how many stations can be lost without impacting operations.

Note: The following calculations in the analytic section will assume that we have 6 wide bore cavities. In the subsequent sections where we discuss ESME simulation results, we will find that we need more than 6.

1. *Loss of legacy cavities*

Let Δ_{lc} be the number of legacy cavities we can afford to lose during PIPII operations. We have to solve the following equation which relates the maximum voltage contributed by the wide bore cavities and the maximum voltage contributed by the legacy cavities *less* the lost legacy cavities:

$$\left.\begin{aligned} N_{wc} \times V_{wc} + (N_{lc} - \Delta_{lc}) \times V_{lc} &= V_{\rm op} \\ \Rightarrow \quad \Delta_{lc} &= N_{lc} - \frac{V_{\rm op} - N_{wc} \times V_{wc}}{V_{lc}} \end{aligned}\right\} \tag{3}$$

Therefore, given the numbers in Table III, we find that

$$\Delta_{lc} = \text{Integer part}\left(16 - \frac{990[\rm kV] - 6 \times 60[\rm kV]}{50[\rm kV]}\right) = 3 \tag{4}$$

i.e. we can afford to lose 3 legacy stations and not impact PIPII operations *provided* that the legacy and wide bore cavities can run at their maximum voltages.

However, if the maximum voltage each legacy cavity can operate is only 45 kV, then

$$\Delta_{lc} = \text{Integer part}\left(16 - \frac{990[\text{kV}] - 6 \times 60[\text{kV}]}{45[\text{kV}]}\right) = 2 \tag{5}$$

i.e. we can afford to lose 2 legacy cavities and still operate, provided the wide bore cavities can operate at 60 kV.

2. Loss of wide bore cavities

In a similar fashion, we can calculate the number of wide bore cavities, Δ_{wc} that we afford to lose without impacting operations. The formula is nearly the same as Eq. 3 but with Δ_{lc} replaced by Δ_{wc}, i.e.

$$\left.\begin{aligned} (N_{wc} - \Delta_{wc}) \times V_{wc} + N_{lc} \times V_{lc} &= V_{\text{op}} \\ \Rightarrow \quad \Delta_{wc} = N_{wc} - \frac{V_{\text{op}} - N_{lc} \times V_{lc}}{V_{wc}} & \end{aligned}\right\} \tag{6}$$

Again, using the numbers in Table III, we find that

$$\Delta_{wc} = \text{Integer part}\left(6 - \frac{990[\text{kV}] - 16 \times 50[\text{kV}]}{60[\text{kV}]}\right) = 2 \tag{7}$$

i.e. we can afford to lose 2 wide bore stations and not impact PIPII operations *provided* that the same stipulations that we have stated for the voltages of the legacy and wide bore cavities hold.

As before, if the legacy cavities can only operate at 45 kV and wide bore cavities at 60 kV, then

$$\Delta_{wc} = \text{Integer part}\left(6 - \frac{990[\text{kV}] - 16 \times 45[\text{kV}]}{60[\text{kV}]}\right) = 1 \tag{8}$$

i.e. we can afford to lose 1 wide bore cavity.

III. BEAM LOADING AND ROBINSON STABILITY

We will divide our analysis into two parts. We will look into whether the beam is stable at p_* and then for the entire frequency ramp from injection to extraction.

A. Stability of the beam at p_*

We will calculate the effect of beam loading at $(dp/dt)_{\max}$. From Eq. 1, $p_* = 5.176$ GeV/c which has a corresponding $\beta(p_*) = 0.984$. This means that $\gamma(p_*) = 5.606$. The transition gamma of Booster is $\gamma_t = 5.45 < \gamma(p_*)$, which means that beam is above transition at p_*. At this point, the revolution frequency is 622.08 kHz and the RF frequency is 52.255 MHz.

1. *Shunt impedance R_s*

In order to calculate the beam loading voltage, we need to know the shunt impedance, R_s at p_* and the component of the beam current at the RF frequency, $I_{\omega_{\mathrm{RF}}}$. We will use the engineer's definition of R_s, i.e. the rms power, P_{rms} required to power a cavity is $P_{\mathrm{rms}} = V_p^2/2R_s$, where V_p is the peak accelerating voltage. The value of R_s at p_* is found by reading the value of R_s at 52.255 MHz on the plot shown in Fig. 12. The shunt impedance is 64.67 kΩ.

2. *Beam current component at ω_{RF}*

The DC current, I_{DC} of N_p protons at p_* is

$$I_{\mathrm{DC}} = qN_p \times f_{\mathrm{rev}}(p_*) = (1.6 \times 10^{-19}[\mathrm{C}]) \times (6.6 \times 10^{12}) \times 621.9[\mathrm{kHz}] = 0.657\ \mathrm{A} \tag{9}$$

For an infinite number of δ-function bunches, the magnitude of the beam current component at ω_{RF} is

$$I_{\omega_{\mathrm{RF}}} = 2 \times I_{\mathrm{DC}} = 1.31\ \mathrm{A} \tag{10}$$

Note: There is a 3 bunch notch out of 84 available buckets, i.e. 81 buckets are filled in Booster. Therefore, there is a small correction to $I_{\omega_{\mathrm{RF}}}$ because of this notch. There is an exact formula for $I_{\omega_{\mathrm{RF}}}$ for an unevenly filled ring that we can use if necessary.[2]

3. *Detuning angle ψ above transition*

The detuning angle ψ above transition is given by the formula[3, 4]

$$\left.\begin{aligned}\psi &= \tan^{-1}\left[\frac{I_{\omega_{\rm RF}}}{V_{lc}/R_s}\cos(\pi-\phi)\right]\\ &= -\tan^{-1}\left[\frac{1.31[\text{A}]}{50[\text{kV}]/64.53[\text{k}\Omega]}\cos\left((130\pi/180\right)\right]\\ &= -\tan^{-1}(1.092) = -0.829\end{aligned}\right\} \tag{11}$$

where we have assumed that the legacy cavities operate at 50 kV. And in degrees $\psi = -47.5°$.

4. *Beam loading voltage V_b*

The beam loading voltage is given by the formula

$$\left.\begin{aligned}V_b &= I_{\omega_{\rm RF}} R_s e^{i\psi}\cos\psi\\ &= 1.31[\text{A}] \times 64.53[\text{k}\Omega] \times e^{-i0.829}\cos 0.829\\ &= (38.7 - 42.3i)\ \text{kV}\end{aligned}\right\} \tag{12}$$

Therefore, the magnitude of the beam loading voltage is $|V_b| = 57.4$ kV.

$|V_b|$ is not small and is larger than the accelerating voltage of each legacy cavity! Therefore, this observation begs the question as to whether the system is Robinson stable. We will consider Robinson stability in the next section.

5. *Robinson stability*

The Robinson stability limit is given by the following inequality[3, 5]

$$\frac{I_{\omega_{\rm RF}}}{I^{\max}_{c||}} < \frac{\cos(\pi-\phi)}{\sin\psi\cos\psi} \qquad \text{above transition, } \psi < 0,. \tag{13}$$

where $I^{\max}_{c||}$ is the cavity current when the generator current is parallel to the cavity voltage. Note: in the literature, this occurs when the load angle $\theta_L = 0$. See ref. [3].

In the case where the legacy cavities are running at their maximum voltage of 50 kV, the value of $I^{\max}_{c||}$ is

$$I^{\max}_{c||} = V_{lc}/R_s = 50[\text{kV}]/64.53[\text{k}\Omega] = 0.77\ \text{A} \tag{14}$$

Using the above inequality, we have

$$\boxed{\frac{I_{\omega_{\rm RF}}}{I_{c||}^{\max}} = \frac{1.31[{\rm A}]}{0.77[{\rm A}]} = 1.70 \not< \frac{\cos(\pi-\phi)}{\sin\psi\cos\psi} = -\frac{\cos(130\pi/180)}{\sin 0.828\cos 0.828} = 1.29} \tag{15}$$

And the stability criterion is *not* satisfied and our system is Robinson unstable. Therefore, beam loading compensation is necessary for PIPII operations.

B. Robinson stability of the beam for the entire frequency ramp

We can discover whether the beam is Robinson stable by plotting the lhs and rhs of Eq. 13 separately. We will assume that all the legacy cavities operate at 50 kV and the accelerating phase ϕ is 50° below transition and 130° above transition.

The detuning curve is shown in Fig. 1.

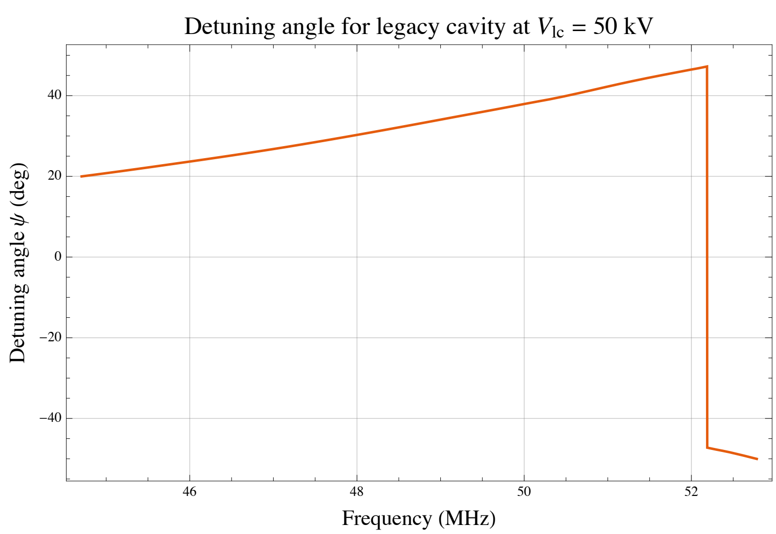

FIG. 1. The detuning angle ψ as a function of RF frequency. Notice the discontinuity that happens at transition when the RF frequency is 52.19 MHz.

The Robinson stability plot for the legacy cavity is shown in Fig. 2. For stability, $I_{\omega_{\rm RF}}/I_{c||}^{\max}$ must be in the shaded region whose upper boundary is $\cos\phi_s/\sin\psi\cos\psi$, where $0 < \phi_s < \pi/2$ below transition and $\pi/2 < \phi_s < \pi$ above transition. From this plot, we can see that the beam is unstable above 50.52 MHz, which corresponds to kinetic energy 2.1 GeV. The beam is still below transition at this point. Note: transition RF frequency is 52.20 MHz and kinetic energy is 4.17 GeV.

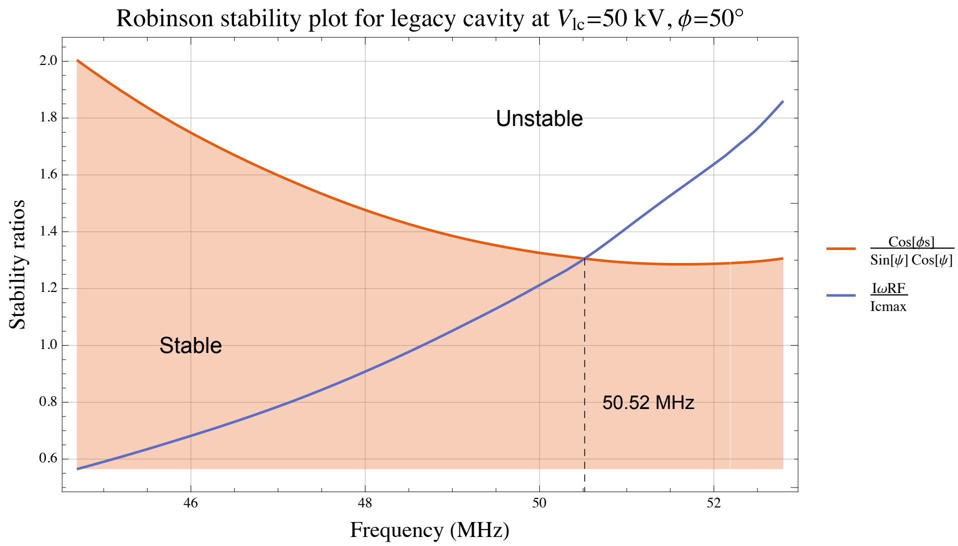

FIG. 2. The Robinson stability of the beam can be read directly from this plot. $I_{\omega_{\mathrm{RF}}}/I_{c||}^{\max}$ must remain in the region shaded in red for the beam to be stable.

IV. ESME SIMULATIONS

The following ESME simulations are for $N_p = 6.6 \times 10^{12}$ protons injected into Booster. Space charge has been turned on. Perfect beam loading compensation is modeled by zeroing out all cavity impedances.

A. Energy ramp

The Booster ramp is modeled as a sinusoid with a flat bottom for injection. The duration of the flat bottom is 750 μs. For our model, we have the flat bottom starting before the minima of the sinusoid. Note: As of 2019, the exact location of flat bottom w.r.t. the sinusoid ramp has not been decided yet. See Fig. 3.

B. RF voltage ramp

The RF voltage ramp used in our simulations is shown in Fig. 4. This ramp has been tuned for the least amount of beam loss from injection to extraction. The model ramp can be compared to the actual ramp used in PIP. We can see that both ramps reach a peak in voltage early in the ramp cycle before it is reduced. Both ramps have similar characteristics.

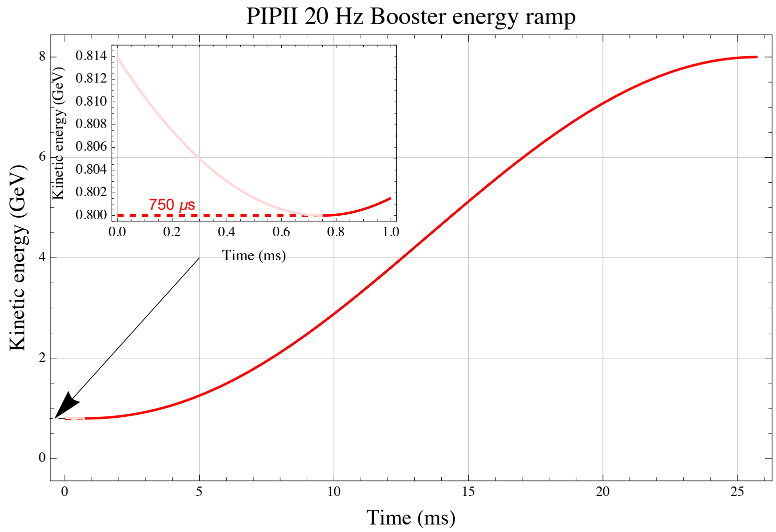

FIG. 3. The 20 Hz Booster energy ramp during the PIPII era. The inset shows our model of the flat injection porch (dashed line) which is 750 μs long and starts before the minimum of the sinusoid. Injection starts at 0 ms.

We have found that the best efficiency for PIPII beam intensities is when the peak voltage is 1.2 MV. This value is larger than the simple back of the envelope voltage, $V_{\text{req}} = 963.4$ kV that was derived in section II.

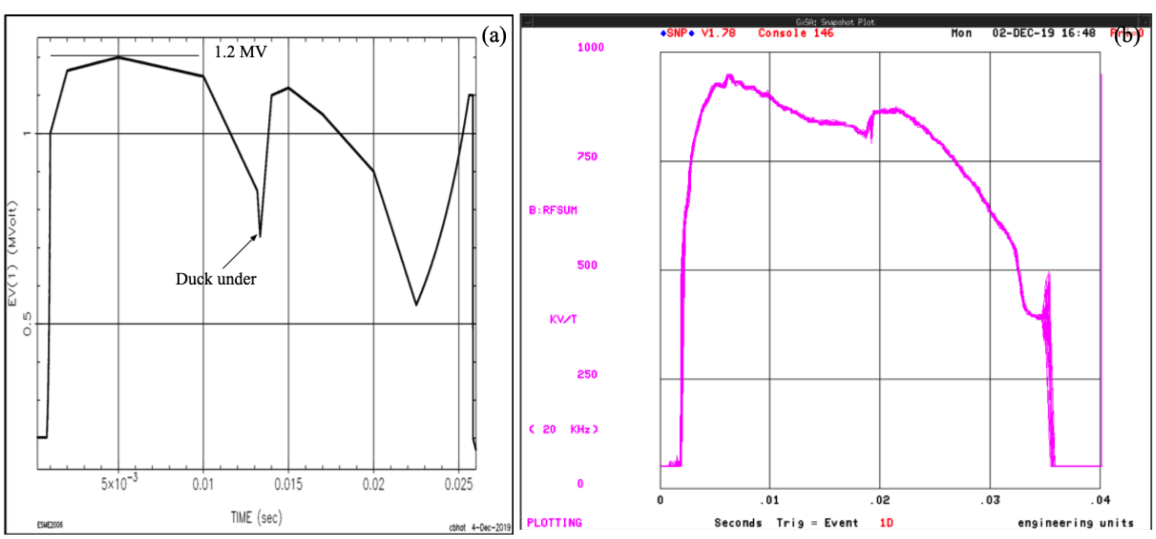

FIG. 4. (a) is the RF voltage ramp used in our ESME simulations. (b) is the RF voltage ramp used in PIP operations.

The PIPII theoretical RF voltage ramp will be discussed in Appendix D. We can compare the ESME ramp shown above to the the theoretical ramp shown in Fig. 14. We find that both ramps show the same behaviour and similar peak voltage of about 1.2 MV.

C. Plots

In this section, we will show the plots from ESME that show the phase space at the interesting points of the energy ramp. The initial distribution of the beam at injection came from P. Derwent.

1. *Injection*

PIPII injection requires painting into the Booster RF bucket to create a doughnut hole in phase space. The beam distribution after injection into Booster and then allowed to evolve to form the doughnut is shown in Fig. 5. The 95% longitudinal emittance is 0.063 eV·s. The RF voltage at this point is 203 kV.

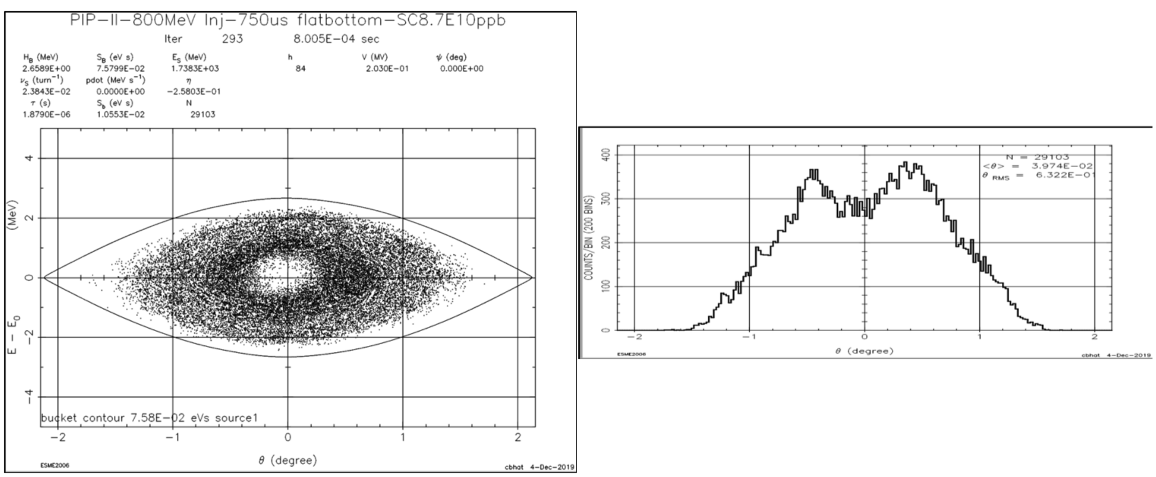

FIG. 5. The injection phase space distribution and line distribuction of the beam at 550 μs into the 750 μs injection porch.

2. *Squeeze*

The beam is "squeezed" at the end of the flat injection porch just before the energy ramp proper. The phase space diagram at 750 μs is shown in Fig. 6. The 95% longitudinal emittance is 0.063 eV·s. The RF voltage at this point is 1 MV.

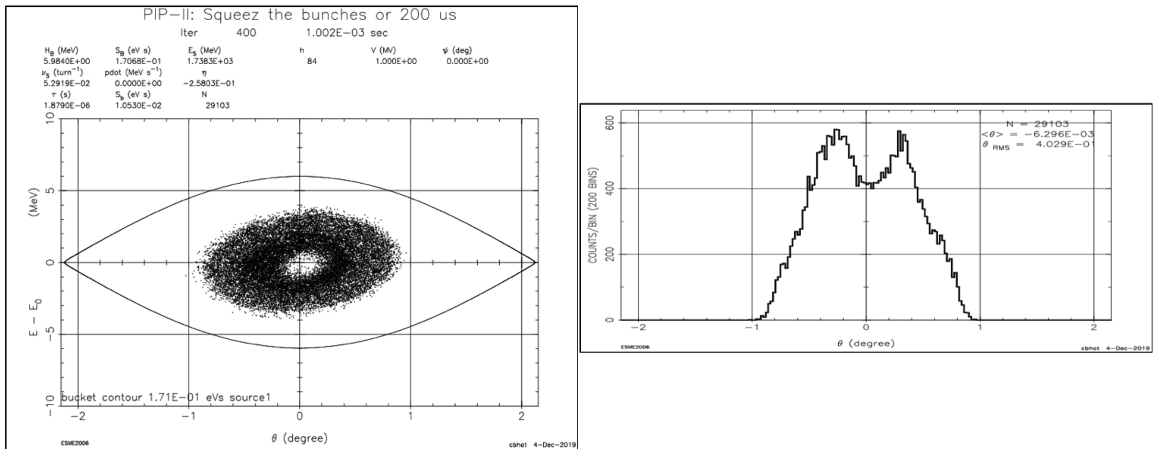

FIG. 6. The phase space distribution and line distribution of the beam at 750 μs just before the start of the energy ramp.

3. Maximum RF voltage

The beam distribution at the maximum RF voltage shown in Fig. 4 is shown Fig. 7. The 95% longitudinal emittance is 0.063 eV·s. The RF voltage at this point is 1.2 MV. The ratio of the bucket area to the beam area is about 2.3 at this point.

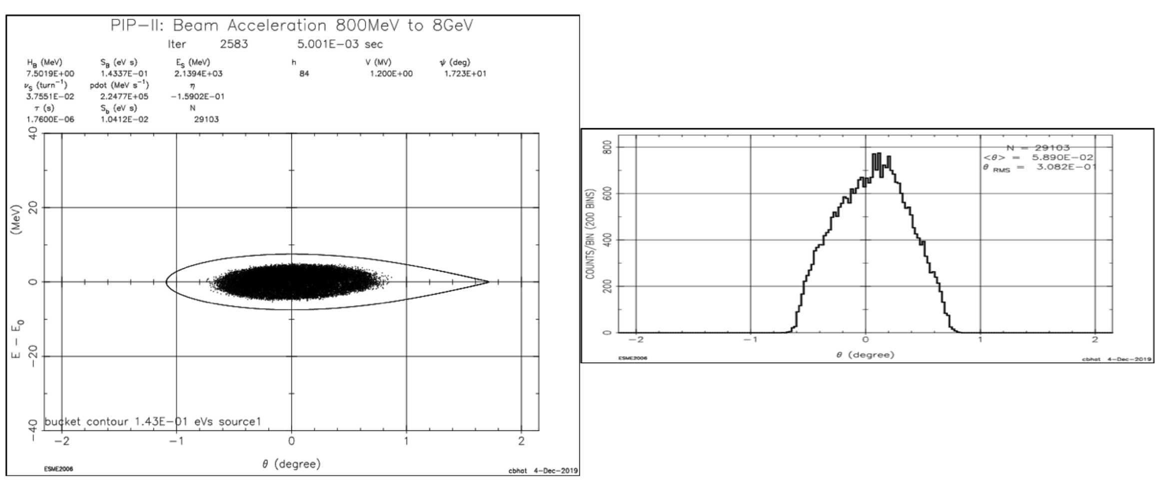

FIG. 7. The phase space distribution and line distribution of the beam at 4.75 ms into the ramp. This is the point where the RF voltage is at its maximum.

4. Transition

The beam just before transition and after transition are shown in Figures 8 and 9. The point just before transition is the marked as “duck under” in Fig. 4(a). The 95% longitudinal emittance is 0.063 eV·s and bunch length is about 2 ns. The “duck under” is needed to reduce the beam energy spread during transition crossing to eliminate beam loss.

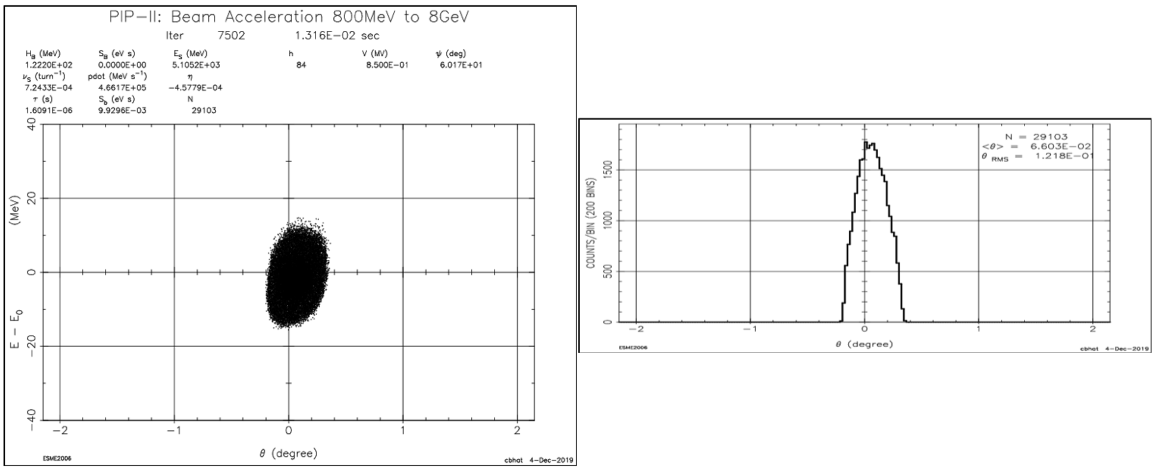

FIG. 8. The beam phase distribution and line distribution Just before transition at 12.95 ms into the energy ramp. The bucket disappears at this point. The RF voltage is 1.2 MV and the bunch length is about 2 ns.

After transition is crossed, the RF voltage is 0.9 MV. The bunch length is about 1.8 ns.

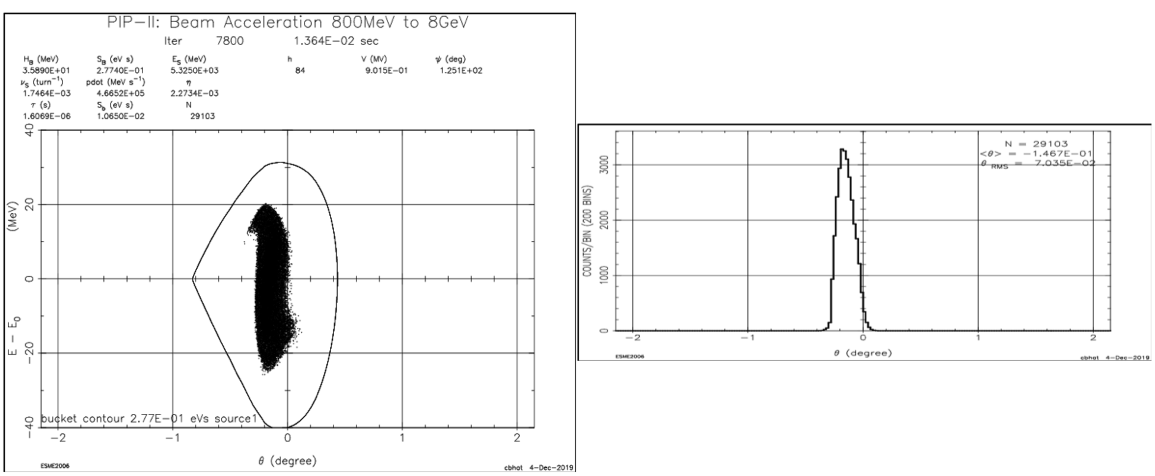

FIG. 9. The beam phase distribution and line distribution Just after transition at 13.39 ms into the energy ramp. The RF voltage is 0.9 MV and the bunch length is about 1.8 ns.

5. End of acceleration

The point just before the end of acceleration at 25.58 ms into the energy ramp is shown in Fig. 10. The 95% longitudinal emittance is $\sim$0.1 eV·s and RF voltage is 1.1 MV and 95% energy spread is $\sim$30 MeV. Subsequently, bunch rotation will be done to reduce the beam energy spread as required by the downstream accelerator.

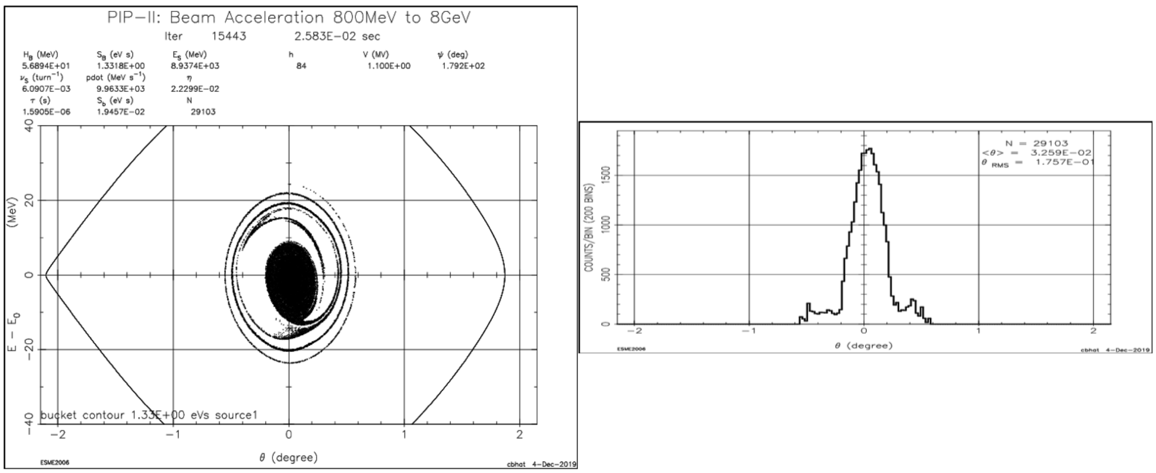

FIG. 10. The beam phase distribution and line distribution near the end of acceleration at 25.58 ms into the energy ramp are shown here.

D. Required cavity mix for 1.2 MV operations without overhead

The ESME results tell us that we need about $V_{\text{req}} = 1.2$ MV for operations in PIPII. It is not possible to reach 1.2 MV with 16 legacy cavities and 6 wide bore cavities when they are operated at 50 kV and 60 kV respectively.

The equation that tells us how many wide bore cavities, N_{wc}, are needed given the total number of cavities $N_c = 22$ and voltages of the legacy cavity V_{lc} and wide bore cavity V_{wc} is

$$\left.\begin{aligned} N_{wc} \times V_{wc} + (N_c - N_{wc}) \times V_{lc} &= V_{\text{req}} \\ \Rightarrow \quad N_{wc} &= \frac{V_{\text{req}} - N_c V_{lc}}{V_{wc} - V_{lc}} \end{aligned}\right\} \tag{16}$$

We cannot operate by equally distributing 1.2 MV to the 22 cavities because the voltage on each cavity will be 54.46 kV which exceeds the proposed voltage on a legacy cavity. The possible ways to combine the legacy and wide bore cavity mixes to achieve 1.2 MV are summarized in Table I.

TABLE I. Number of legacy and wide bore cavities required *without* overhead

Scenario	**Legacy cavity V_{lc}(kV)**	**Wide bore cavity V_{wc} (kV)**	**# Legacy cavity**	**# Wide bore cavity**
1	45	60	8	14
2	50	60	12	10

Unfortunately, these two scenarios do not allow us to operate with any overhead. Any cavity failure will reduce the available RF voltage without any possibility of compensation with the remaining working cavities.

E. Required cavity mix for 1.2 MV operations with overhead

We will suppose that we require normal operations at full beam intensity of 6.6×10^{12} protons per batch when we lose one cavity. This cavity can either be a legacy or a wide bore cavity. In order to have overhead, we must have more wide bore cavities than those shown in Table I. These additional wide bore cavities will operate at the legacy cavity voltages, V_{lc}, so that when any one cavity is down, their voltage can be increased to cover for the downed cavity.

The formula for calculating the number of extra wide bore cavities, n_{wc}, is trivial and it is

$$(N_{wc} + n_{wc})V_{wc} + (\tilde{N}_{lc} - n_b - n_{wc})V_{lc} = V_{\text{req}} \quad \text{if there are } n_b \text{ broken cavities operating at } V_{lc} \tag{17}$$

$$(N_{wc} - n_b + n_{wc})V_{wc} + (\tilde{N}_{lc} - n_{wc})V_{lc} = V_{\text{req}} \quad \text{if there are } n_b \text{ broken cavities operating at } V_{wc} \tag{18}$$

where $\tilde{N}_{lc} = N_{lc} + n_{wc}$, i.e. the total number of cavities operating at the legacy cavity accelerating voltage, V_{lc}, is the sum of the number of actual legacy cavities and the number of wide bore cavities operating at this lower voltage.

Using the above formulæ, we can fill in Table II for the two scenarios described in Table I. We will assume that when there is a broken cavity, *all* the wide bore cavities running at the lower legacy voltage, V_{lc}, will be increased to V_{wc} to compensate.

TABLE II. Number of legacy and wide bore cavities required *with* overhead for 1 broken cavity

Scenario	Legacy V_{lc}(kV)	Wide bore V_{wc} (kV)	Broken type	at V_{lc} # Legacy	at V_{lc} # Wide bore	at V_{wc} # Wide bore	Total # wide bore
1a	45	60	Legacy	5	3	14	17
1b	45	60	Wide bore	4	4	14	18
2a	50	60	Legacy	7	5	10	15
2b	50	60	Wide bore	6	6	10	16

For example, in the most optimistic case where the legacy cavities can indeed operate at 50 kV (scenario 2b), when we lose one wide bore cavity, we need 16 wide bore cavities, all operating at $V_{wc} = 60$ kV to compensate.

V. CONCLUSION

Our calculations show that beam loading compensation is required because the beam will be Robinson unstable without it when the beam is above 2 GeV. ESME simulations tell us that for successfully accelerating 6.6×10^{12} protons with perfect beam loading compensation and zero beam loss is about 1.2 MV. This value also assumes that the cavities are in phase and do not have any errors. The lowest minimum number of wide bore cavities that allow us to achieve this voltage is 16 with the allowance that we can have one failed wide bore cavity. However, this scenario assumes that the legacy cavities can operate reliably at 50 kV and the wide bore cavities can operate at 60 kV. Again, these numbers are given under the caveats given in section I A.

Appendix A: Required minimum accelerating voltage without space charge

Note: This calculation here assumes a δ-function beam. For the case when there is a finite longitudinal emittance, the method discussed in Appendix D is more appropriate.

The required minimum accelerated voltage *without* any beam loading considerations is derived here. We will assume that Booster has a sinusoidal ramp with frequency ω_{ramp}. The momentum, p, of the beam varies as follows:

$$p(t) = \frac{p_f + p_i}{2} - \frac{p_f - p_i}{2}\cos\omega_{\text{ramp}}t \tag{A1}$$

where p_i and p_f are the injection and extraction momenta of the beam.

We want to find the maximum value of dp/dt. This value will determine the required accelerating voltage. To do this, we differentiate $p(t)$ once to get

$$\frac{dp}{dt} = \frac{p_f - p_i}{2}\omega_{\text{ramp}}\sin\omega_{\text{ramp}}t \tag{A2}$$

from which it is obvious that dp/dt is maximum when $\omega_{\text{ramp}}t = \pi/2$, i.e.

$$\left(\frac{dp}{dt}\right)_{\text{max}} = \omega_{\text{ramp}}\frac{p_f - p_i}{2} \tag{A3}$$

For the required accelerating voltage to give at least $(dp/dt)_{\text{max}}$, we note that the energy, E, gained per proton per turn (assuming that we have a δ-function bunch) is

$$\frac{dE}{dn} = qV\sin\phi \tag{A4}$$

where q is the electronic charge, V is the voltage of the RF cavity and ϕ is the synchronous phase.

But $E^2 = p^2 + m^2$ and so

$$\frac{dE}{dn} = \frac{p}{E}\frac{dp}{dn} \tag{A5}$$

which we can substitute into A4 to get

$$\left.\begin{aligned} \frac{p}{E}\frac{dp}{dn} &= qV\sin\phi \\ \frac{p}{E}\frac{1}{f_{\text{rev}}(p)}\frac{dp}{dt} &= qV\sin\phi \end{aligned}\right\} \tag{A6}$$

where we have used $dn = dt/T_{\text{rev}}$ with $T_{\text{rev}} = 1/f_{\text{rev}}$ is the revolution period at momentum p.

The required accelerating voltage, V_{req} when $(dp/dt)_{\text{max}}$ is thus

$$\frac{p_*}{E_*}\frac{1}{f_{\text{rev}}(p_*)}\left(\frac{dp}{dt}\right)_{\text{max}} = qV_{\text{req}}\sin\phi \tag{A7}$$

where p_* is the momentum when $\omega_{\text{ramp}}t = \pi/2$ (see Eq. A3). Therefore, we have from Eq. A1

$$p_* = \frac{p_f + p_i}{2} \tag{A8}$$

We can solve for V_{req} given $(dp/dt)_{\max}$ from Eq. A3, p_* from Eq. A8 and $E_* = \sqrt{p_*^2 + m^2}$ where m is the mass of the proton in eV/c^2. Note: dimensions of E_* is in eV and p_* is in eV/c, to get the required voltage

$$\boxed{V_{\text{req}} = \frac{f_{\text{ramp}}(p_f^2 - p_i^2)\pi}{f_{\text{rev}}(p_*)\left[4m^2 + (p_f + p_i)^2\right]^{1/2}\sin\phi}} \tag{A9}$$

where the charge q is not in the result because we are working in eV type units. And $f_{\text{rev}}(p_*)$ is easily found because we know the the radius of Booster and p_*.

Appendix B: Phase error and reduction in acceleration voltage

Let us assume that the phase error between cavity 1 and cavity j is $\Delta\theta_j \ll 1$. Then the vector sum of voltages, V_c of the N_c cavities is simply

$$V_c = V_p\left(1 + \sum_{j=2}^{N_c} e^{i\Delta\theta_j}\right) \tag{B1}$$

We need the magnitude of the total cavity voltage, $|V_c|$, which is easily found by first finding the square of the magnitude of V_c which is

$$\begin{aligned} |V_c|^2 &= V_p^2\left(1 + \sum_{j=2}^{N_c} e^{i\Delta\theta_j}\right)\left(1 + \sum_{j=2}^{N_c} e^{-i\Delta\theta_j}\right) \\ &= V_p^2\left(1 + \sum_{j=2}^{N_c} e^{i\Delta\theta_j} + \sum_{j=2}^{N_c} e^{-i\Delta\theta_j} + \sum_{j=2}^{N_c}\sum_{k=2}^{N_c} e^{i(\Delta\theta_j - \Delta\theta_k)}\right) \end{aligned} \tag{B2}$$

where V_p is the voltage contribution from one RF cavity and is assumed to be the same for each of the N_c cavities. At this point, we cannot go anywhere with the above unless we perform statistical analysis on it.

Let us suppose that we have, on average, $\langle\Delta\theta_j\rangle = 0$ and variance $\langle\Delta\theta_j^2\rangle \equiv \langle\Delta\theta_j^2\rangle = \sigma^2$ then Eq. B2 becomes

$$\langle|V_c|^2\rangle = V_p^2\left(1 + \left\langle\sum_{j=2}^{N_c} e^{i\Delta\theta_j}\right\rangle + \left\langle\sum_{j=2}^{N_c} e^{-i\Delta\theta_j}\right\rangle + \left\langle\sum_{j=2}^{N_c}\sum_{k=2}^{N_c} e^{i(\Delta\theta_j - \Delta\theta_k)}\right\rangle\right) \tag{B3}$$

which we then deal with each of the sums on the rhs separately. First

$$\begin{aligned}\left\langle\sum_{j=2}^{N_c} e^{i\Delta\theta_j}\right\rangle+\left\langle\sum_{j=2}^{N_c} e^{-i\Delta\theta_j}\right\rangle &= \left\langle\sum_{j=2}^{N_c}(\cos\Delta\theta_j+i\sin\Delta\theta_j)\right\rangle+\left\langle\sum_{j=2}^{N_c}(\cos\Delta\theta_j-i\sin\Delta\theta_j)\right\rangle\\ &\approx \left\langle\sum_{j=2}^{N_c}(1-\frac{1}{2}\Delta\theta_j^2+i\Delta\theta_j)\right\rangle+\left\langle\sum_{j=2}^{N_c}(1-\frac{1}{2}\Delta\theta_j^2-i\Delta\theta_j)\right\rangle\\ &= 2(N_c-1)-(N_c-1)\sigma^2\end{aligned} \tag{B4}$$

Second

$$\left.\begin{aligned}\left\langle\sum_{j=2}^{N_c}\sum_{k=2}^{N_c} e^{i(\Delta\theta_j-\Delta\theta_k)}\right\rangle &= \left\langle\sum_{j=2}^{N_c}\sum_{k=2}^{N_c}\cos(\Delta\theta_j-\Delta\theta_k)+i\sin(\Delta\theta_j-\Delta\theta_k)\right\rangle\\ &\approx \left\langle\sum_{j=2}^{N_c}\sum_{k=2}^{N_c}\left(1-\frac{1}{2}\Delta\theta_j^2\right)\left(1-\frac{1}{2}\Delta\theta_k^2\right)+\Delta\theta_j\Delta\theta_k\right.\\ &\quad\left. +i\left[\Delta\theta_j\left(1-\frac{1}{2}\Delta\theta_k^2\right)-\Delta\theta_k\left(1-\frac{1}{2}\Delta\theta_j^2\right)\right]\right\rangle\\ &= (N_c-1)^2-(N_c-1)^2\sigma^2\end{aligned}\right\} \tag{B5}$$

because the cross terms like $\sum_j\sum_k\langle\Delta\theta_j\Delta\theta_k\rangle = 0$. (Note: easy to show because terms like $\Delta\theta_j\langle\Delta\theta_k\rangle = 0$).

Therefore, Eq. B3 becomes

$$\left.\begin{aligned}\langle|V_c|^2\rangle &= V_p^2[(1+2(N_c-1)-(N_c-1)\sigma^2+(N_c-1)^2-(N_c-1)^2\sigma^2]\\ &= V_p^2[N_c^2-N_c(N_c-1)\sigma^2]\end{aligned}\right\} \tag{B6}$$

Finally, this is the formula that we are seeking for the rms cavity voltage, V_c, given the rms phase error σ and it is

$$\boxed{\sqrt{\langle|V_c|^2\rangle} = V_p\sqrt{N_c^2-N_c(N_c-1)\sigma^2}} \tag{B7}$$

which, of course, becomes $|V_c| = N_cV_p$ when there is no phase error.

Appendix C: Parameters

The parameters used in this writeup are shown in Table III.

TABLE III. PIPII Booster parameters

Parameter	**Variable**	**Value**	**Units**
mass of proton	m	0.9383	GeV/c^2
speed of light	c	2.9979458×10^8	m/s
proton charge	q	1.6×10^{-19}	C
number of protons	N_p	6.6×10^{12}	
harmonic number	h	84	
radius of Booster	R	75.47	m
ramp frequency	f_{ramp}	20	Hz
acceleration synchronous phase	ϕ	50[a]	degrees
injection kinetic energy	E_i	0.8	GeV
injection momentum	p_i	1.463	GeV/c
transition kinetic energy	E_t	4.18	GeV
transition momentum	p_t	5.03	GeV/c
transition γ	γ_t	5.45	
extraction kinetic energy	E_f	8	GeV
extraction momentum	p_f	8.889	GeV/c
total number of cavities	$N_c = N_{wc} + N_{lc}$	22	
number of wide bore cavities	N_{wc}	–[b]	
maximum voltage of wide bore cavities	V_{wc}	60	kV
number of legacy cavities	N_{lc}	$N_c - N_{wc}$	
maximum voltage of legacy cavities	V_{lc}	50	kV

[a] This value is given for below transition. Above transition the synchronous phase is $(180 - 50)^\circ = 130^\circ$. This value is not used in Appendix D.

[b] The number of wide bore cavities varies depending on the operating scenarios. See each section above for the discussion.

Notes:

1. The proposed number of wide bore cavities as of Dec 2019 is $N_{wc} = 6$.

2. The measured Q and shunt impedance R_s of a legacy cavity as a function of RF frequency are shown in Figures 11 and 12. Note: the shunt impedance is the engineer's definition, i.e. $P_{\text{rms}} = V_p^2/2R_s$, where P_{rms} is the rms power required to power the cavity and V_p is the peak voltage in the gap of the cavity.

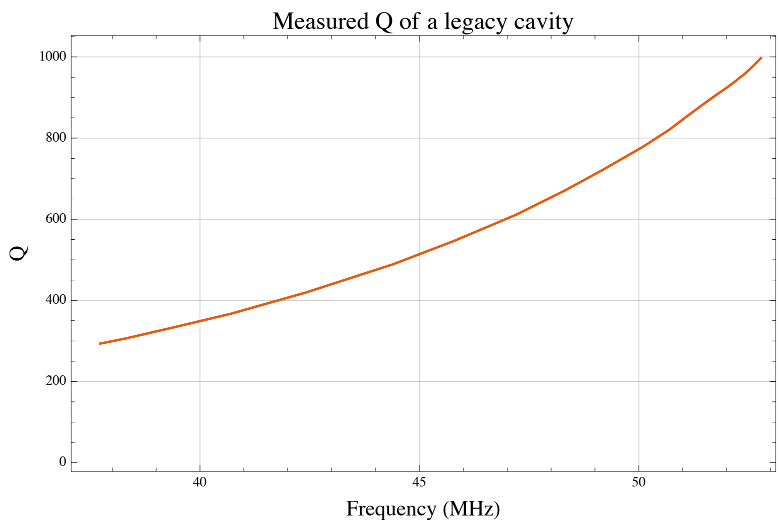

FIG. 11. The measured Q of a legacy cavity.

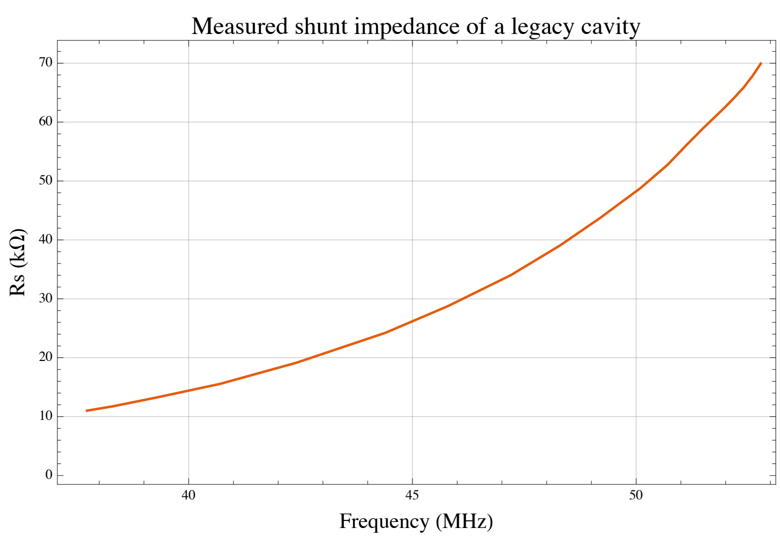

FIG. 12. The measured shunt impedance of a legacy cavity.

Appendix D: Theoretical RF voltage curve

This work comes from P. Derwent. The inspiration for the derivation came from Griffin fig. 5.[5]

The theoretical RF voltage curve can be found by making the assumption that the bucket has to hold the longitudinal phase space occupied by the beam. When we are given the longitudinal emittance ϵ in eV·s, we can set up the required bucket size

$$\varepsilon = \Lambda\epsilon \tag{D1}$$

where Λ is the bucket to emittance ratio that ensures that there is no beam loss.

The bucket area, B of an accelerating or moving bucket is given by

$$B = \frac{16R}{hc}\sqrt{\frac{EV}{2\pi h|\eta|}}\alpha(\Gamma) \tag{D2}$$

where $\eta = 1/\gamma^2 - 1/\gamma_t^2$ is the slip factor, E is the total energy of the beam, V is the peak of the RF voltage, and α is the factor that reduces the area of the stationary bucket due to the accelerating phase ϕ and $\Gamma = \sin\phi$. We use the following parametrization of α from Lebedev

$$\alpha(\Gamma) = \begin{cases} \dfrac{1-\Gamma}{(1+\Gamma/2)^2} & \text{if } \Gamma < 1 \\ 1\times 10^{-10} & \text{if } \Gamma \geq 1 \end{cases} \tag{D3}$$

We can invert the above to find the required peak voltage, V, for a given bucket size $B = \varepsilon$

$$V = \frac{2\pi h|\eta|}{E}\left(\frac{hc\varepsilon}{16R\alpha(\Gamma)}\right)^2 \tag{D4}$$

which means that the accelerating voltage seen by the synchronous particle is simply $V\sin\phi = V\Gamma$.

Unlike the previous sections, we will *not* assume that the accelerating phase, $\phi = 50°$ or $130°$ depending on where the beam is w.r.t. transition. We will derive the synchronous phase from Eq. D4 and the required minimum RF voltage, V_{min}, for accelerating the beam given the momentum ramp:

$$p(t) = \frac{p_f + p_i}{2} - \frac{p_f - p_i}{2}\cos\omega_{\text{ramp}}t \tag{D5}$$

Note: this equation does not taken into account the 750 ms flat portion of the injection ramp shown in Fig. 3.

From the above, we can calculate the total energy $E(t) = \sqrt{p(t)^2 + m^2}$, and the minimum accelerating voltage, V_{min}, that takes the beam from E_i to E_j for the above ramp. V_{min} comes from Eq. A6 when $\phi = \pi/2$, i.e. the proton is riding at the crest of the RF

$$V_{\text{min}} = \frac{1}{q}\frac{p}{E}\frac{1}{f_{\text{rev}}(p)}\frac{dp}{dt} \tag{D6}$$

with dp/dt coming from Eq. A2.

Finally, we can solve for Γ using

$$\left.\begin{aligned} V_{\text{min}} &= V\Gamma \\ \Rightarrow \quad \frac{1}{q}\frac{p}{E}\frac{1}{f_{\text{rev}}(p)}\frac{dp}{dt} &= \frac{2\pi h|\eta|}{E}\left(\frac{hc\varepsilon}{16R\alpha(\Gamma)}\right)^2 \Gamma \end{aligned}\right\} \tag{D7}$$

with Eq. D4 and D6 to obtain the curve shown in Fig. 13 for $\Lambda = 2.3$ and $\epsilon = 0.063$ eV·s obtained from our ESME simulation in section IV C 1.

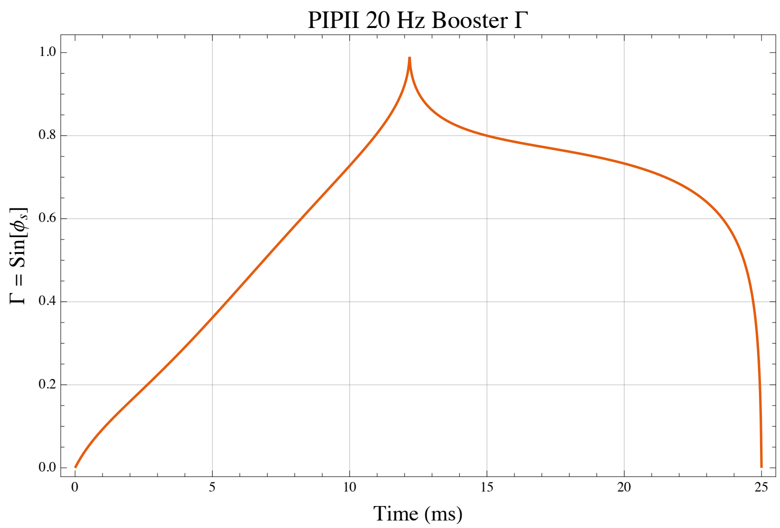

FIG. 13. Γ that was found by solving Eq. D7 for $\varepsilon = (2.3 \times 0.063)$ eV·s.

Thus, the required peak accelerating voltage V for the Γ shown in Fig. 13 can be found using Eq. D5. This gives us the red curve shown in Fig. 14. The maximum RF voltage found by this method is 1.225 MV at 3.699 ms. These numbers agree with ESME that we need about 1.2 MV to keep the beam within the bucket for the entire acceleration ramp.

Here are a few interesting notes:

1. This simple calculation produces an RF voltage curve that looks similar to the ESME RF voltage curve before transition. This observation tells us that voltage curve, at

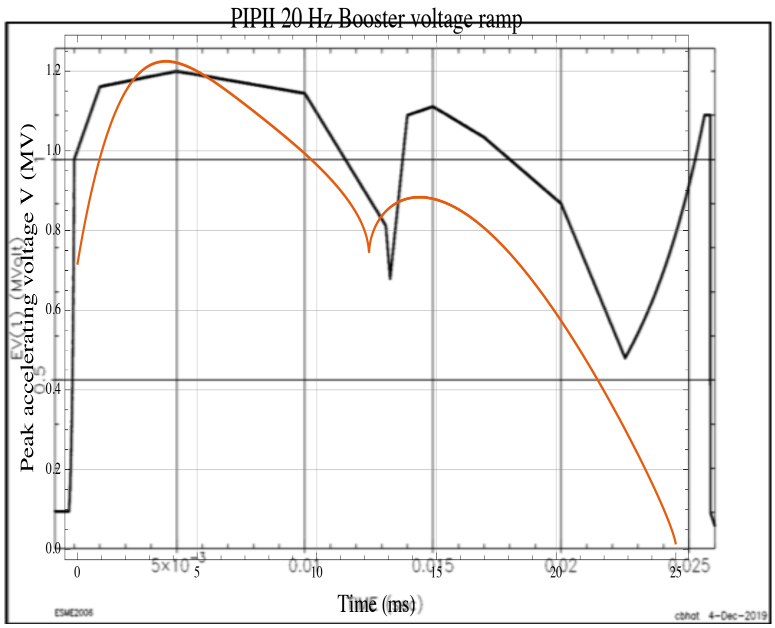

FIG. 14. The theoretical voltage ramp (red) using Γ that we found in Fig. 13. We have superimposed the ESME ramp shown in Fig. 4. Both ramps reach a value of about 1.2 MV.

least before transition, is solely determined by the initial longitudinal emittance that was "blown up" by space charge.

2. The maximum RF voltage is *not* at where dp/dt has its maximum value. $(dp/dt)_{\rm max}$ occurs above transition around $\pi/(2\omega_{\rm ramp}) = 125$ ms (because we have not taken into account the 750 ms flat part of the injection ramp). The reason why the location of the peak calculated here is different from that calculated in Appendix A is because the beam has a longitudinal emittance.

3. This calculation does not include space charge or the number of protons. It only uses the size of the bucket ε to ensure that the beam is contained for the entire ramp.

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